COMPACTNESS OF THE $\overline{\partial}$-NEUMANN OPERATOR AND COMMUTATORS OF THE BERGMAN PROJECTION WITH CONTINUOUS FUNCTIONS

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ABSTRACT. Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n, n \geq 2, 0 \leq p \leq n$, and $1 \leq q \leq n-1$. We show that compactness of the $\overline{\partial}$-Neumann operator, $N_{p,q+1}$, on square integrable $(p,q+1)$-forms is equivalent to compactness of the commutators $[P_{p,q}, \overline{z}_j]$ on square integrable $\overline{\partial}$-closed (p,q)-forms for $1 \leq j \leq n$ where $P_{p,q}$ is the Bergman projection on (p,q)-forms. We also show that compactness of the commutator of the Bergman projection with functions continuous on $\overline{\Omega}$ percolates up in the $\overline{\partial}$-complex on $\overline{\partial}$-closed forms and square integrable holomorphic forms.

The purpose of this paper is to characterize compactness of the $\overline{\partial}$-Neumann operator on square integrable (p,q)-forms. Theorem 1 provides six equivalent statements, for q at least 2, on bounded pseudoconvex domains. However, the important special case of functions, namely $(0,0)$-forms, remains open. In Remark 3 we discuss why our proof breaks down in this case.

Compactness results in the $\overline{\partial}$-Neumann problem have a long history; we refer to a recent book by Straube [Str10] for a detailed discussion. We note here that most known results provide conditions for compactness in terms of the boundary geometry. It is also useful to characterize compactness in functional analytic terms. For example, Catlin and D'Angelo [CD97, Theorem 1] used compactness of the commutators $[P,\phi]$ of the Bergman projection P and certain multiplication operators ϕ in conjunction with a complex variables analogue of Hilbert's 17th problem (see also [D'A11]). In the same paper they showed that compactness of $N_{0,1}$ implies that the commutators $[P,M]$ are compact for all tangential pseudodifferential operators M of order 0. D'Angelo then asked the following question:

Question 1. *Can one characterize compactness of the $\overline{\partial}$-Neumann operator in terms of commutators $[P,\phi]$?*

This question is appealing because of its connection to operator theory as well. The Hankel operator, H_ϕ, with a bounded symbol ϕ is defined as $H_\phi(f) = (I-P)(\phi f)$ where f is a square integrable holomorphic function. Using Kohn's formula, $P = I - \overline{\partial}^* N_{0,1}\overline{\partial}$, one can compute that $H_\phi f = \overline{\partial}^* N_{0,1}\overline{\partial}(f\phi)$. Using this formula Čučković and Şahutoğlu [ČŞ09] studied how boundary geometry interacts with Hankel operators. They showed that, on smooth bounded convex domains in $\mathbb{C}^2$, compactness of H_ϕ can be characterized by the behavior of ϕ on analytic discs

Date: November 22, 2012.
2010 *Mathematics Subject Classification.* Primary 32W05; Secondary 46B35.
Key words and phrases. $\overline{\partial}$-Neumann operator, Hankel operators, Bergman projection, pseudoconvex domain.

in the boundary. Here ϕ is smooth up to the boundary. It would be interesting to know if this characterization still holds in higher dimensions.

On convex domains the relation between the compactness of the commutators and of the $\overline{\partial}$-Neumann operator has been fairly well understood: if Ω is a bounded convex domain, then compactness of $N_{p,q+1}$ is equivalent to compactness of the commutators $[P_{p,q}, \phi]$ on the space of (p,q)-forms with square integrable holomorphic coefficients, for all functions ϕ continuous on $\overline{\Omega}$ (see [Str10, Remark (ii) in Section 4.1]).

In our previous paper [ÇŞ12] we answered Question 1 on non-pseudoconvex domains. To be more precise, we constructed a smooth bounded non-pseudoconvex domain in $\mathbb{C}^n$ (for a given $n \geq 3$) for which the commutators $[P, \phi]$ are compact (on square integrable functions) for all ϕ continuous on the closure of the domain yet the $\overline{\partial}$-Neumann operator $N_{0,1}$ is not compact. In this paper we will consider the important pseudoconvex case and establish a decisive result on forms in Theorem 1.

We would like to thank Željko Čučković, John D'Angelo, and Emil Straube for valuable comments on a preliminary version of this manuscript.

1. Background and Main Results

Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$.We denote the square integrable (p,q)-forms and the $\overline{\partial}$-closed (p,q)-forms on Ω by $L^2_{(p,q)}(\Omega)$ and $K^2_{(p,q)}(\Omega)$, respectively. The $\overline{\partial}$-Neumann operator, $N_{p,q}$, is defined as the solution operator for $\Box_{p,q} u = v$ where $\Box_{p,q} = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$ on $L^2_{(p,q)}(\Omega)$ and $\overline{\partial}^*$ is the Hilbert space adjoint of $\overline{\partial}$. Hörmander [Hör65] showed that $N_{p,q}$ is a bounded operator when Ω is a bounded pseudoconvex domain. Kohn in [Koh63] connected the Bergman projection $P_{p,q}$ on (p,q)-forms with the $\overline{\partial}$-Neumann operator by the formula

$$P_{p,q} = I - \overline{\partial}^* N_{p,q+1}\overline{\partial}.$$

We note that $P_{p,q}$ is the orthogonal projection onto $K^2_{(p,q)}(\Omega)$, the operator $P_{0,0}$ is the classical Bergman projection P, and $K^2_{(0,0)}(\Omega)$ (also denoted as $A^2(\Omega)$) is called the Bergman space. We refer the reader to [CS01, Str10] for more information about the $\overline{\partial}$-Neumann problem and related issues.

The commutators of the Bergman projection with multiplication operators can also be written in terms of the $\overline{\partial}$-Neumann operator. One can use Kohn's formula to compute that

$$[P_{p,q}, \phi]f = -\overline{\partial}^* N_{p,q+1}\overline{\partial}\phi \wedge f$$

for $f \in K^2_{(p,q)}(\Omega)$ and $\phi \in C^1(\overline{\Omega})$. Furthermore, compactness of $N_{p,q+1}$ on $L^2_{(p,q+1)}(\Omega)$ implies that $[P_{p,q}, \phi]$ is compact on $L^2_{(p,q)}(\Omega)$ for all $\phi \in C(\overline{\Omega})$ (see [Str10, Propositions 4.1 and 4.2]). In the following theorem we show that the the converse is true when $q \geq 1$.

Theorem 1. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n, n \geq 2, 0 \leq p \leq n$, and $1 \leq q \leq n-1$. Then the following are equivalent:*

i. $N_{p,q+1}$ *is compact on* $L^2_{(p,q+1)}(\Omega)$,
ii. $\overline{\partial}^* N_{p,q+1}$ *is compact on* $K^2_{(p,q+1)}(\Omega)$,
iii. $[P_{p,q}, \overline{z}_j]$ *is compact on* $K^2_{(p,q)}(\Omega)$ *for all* $1 \leq j \leq n$,
iv. $[P_{p,q}, \overline{z}_j]$ *is compact on* $L^2_{(p,q)}(\Omega)$ *for all* $1 \leq j \leq n$,
v. $[P_{p,q}, \phi]$ *is compact on* $L^2_{(p,q)}(\Omega)$ *for all* $\phi \in C(\overline{\Omega})$,
vi. $[P_{p,q}, \phi]$ *is compact on* $K^2_{(p,q)}(\Omega)$ *for all* $\phi \in C(\overline{\Omega})$.

We would like to mention that Haslinger in [Has08, Theorem 3] proved equivalence of iii., iv., v., and vi. in Theorem 1 in case of $p = q = 0$.

One consequence of Theorem 1 is the following observation: To conclude that $N_{p,q}$ is compact it is enough to verify compactness of $\overline{\partial}^* N_{p,q}$ only, instead of verifying compactness of both $\overline{\partial}^* N_{p,q}$ and $\overline{\partial}^* N_{p,q+1}$ (see [Str10, Proposition 4.2]).

Remark 1. We note that compactness of $N_{p,0}$ on the orthogonal complement of $A^2(\Omega)$ is equivalent to compactness of $N_{p,1}$ on $L^2_{(p,1)}(\Omega)$. This can be seen as follows: $N_{p,0} = \left(\overline{\partial}^* N_{p,1}\right)\left(\overline{\partial}^* N_{p,1}\right)^*$. This formula shows that compactness of $N_{p,0}$ implies compactness of $\overline{\partial}^* N_{p,1}$. Then Lemma 3 implies that $\overline{\partial}^* N_{p,2}$ compact. Finally, Range's Formula (see [Str10, p. 77] and [Ran84]) for $q = 1$

$$N_{p,1} = \left(\overline{\partial}^* N_{p,1}\right)^* \left(\overline{\partial}^* N_{p,1}\right) + \left(\overline{\partial}^* N_{p,2}\right)\left(\overline{\partial}^* N_{p,2}\right)^*$$

implies that $N_{p,1}$ is compact. On the other hand, compactness $N_{p,1}$ implies compactness of $\overline{\partial}^* N_{p,1}$. In turn, the formula $N_{p,0} = \left(\overline{\partial}^* N_{p,1}\right)\left(\overline{\partial}^* N_{p,1}\right)^*$ shows that, in this case, $N_{p,0}$ is compact on the orthogonal complement of $A^2(\Omega)$.

Compactness of the $\overline{\partial}$-Neumann operator percolates up in the $\overline{\partial}$-complex (see Remark 6). The following theorem shows that the same is true for the commutator of the Bergman projection with a function continuous on the closure of the domain.

Theorem 2. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ for $n \geq 2, 0 \leq p \leq n, 1 \leq q \leq n-1$, and $\phi \in C(\overline{\Omega})$. Assume that $[P_{p,q}, \phi]$ is compact on $K^2_{(p,q)}(\Omega)$. Then $[P_{p,q+1}, \phi]$ is compact on $K^2_{(p,q+1)}(\Omega)$.*

Remark 2. Compactness of $[P_{p,q}, \phi]$ on $K^2_{(p,q)}(\Omega)$ does not necessarily imply compactness of $[P_{p,q}, \phi]$ on $L^2_{(p,q)}(\Omega)$, in general. One can construct a counterexample as follows: Let $\Omega \subset \mathbb{C}^n$ be the polydisk, $0 \leq p \leq n, 0 \leq q \leq n-2$, and consider $\phi = z_j$ for $1 \leq j \leq n$. Then $[P_{p,q}, z_j] = 0$ on $K^2_{(p,q)}(\Omega)$, hence compact. On the other hand, if $[P_{p,q}, z_j]$ were compact on $L^2_{(p,q)}(\Omega)$ the adjoint $[P_{p,q}, z_j]^* = -[P_{p,q}, \overline{z}_j]$ would be compact on $L^2_{(p,q)}(\Omega)$ as well. We note that

$$-[P_{p,q}, \overline{z}_j]g = \overline{\partial}^* N_{pq+1}(d\overline{z}_j \wedge g)$$

for $g \in K^2_{(p,q)}(\Omega)$. In addition, one can show that any $f \in A^2_{(p,q+1)}(\Omega)$ can be written as $f = \sum_{j=1}^n f_j \wedge d\overline{z}_j$ where $f_j \in A^2_{(p,q)}(\Omega)$ and $\sum_{j=1}^n \|f_j\|^2 = \|f\|^2$ (see the proof of Corollary 2 in the next section). Then we have

$$\overline{\partial}^* N_{p,q+1} f = (-1)^{p+q+1} \sum_{j=1}^n [P_{p,q}, \overline{z}_j] f_j.$$

Therefore, if $[P_{p,q}, z_j]$ were compact on $L^2_{(p,q)}(\Omega)$ for $1 \leq j \leq n$ the operator $\overline{\partial}^* N_{p,q+1}$ would be compact on $A^2_{(p,q+1)}(\Omega)$. However, this contradicts [FS98, Theorem 1.1]. We note that even though [FS98, Theorem 1.1] is stated on $L^2_{(0,q)}(\Omega)$ the proof only uses forms with holomorphic coefficients and the proof is valid on (p,q)-forms as well (see remarks 2 and 4 in [FS98, pg. 638]).

2. PROOFS OF THEOREM 1 AND 2

The proof of Theorem 1 will be based on several lemmas whose proofs are elementary if one is familiar with the basics of the $\overline{\partial}$-Neumann problem.

Lemma 1. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ for $n \geq 2$ and $g \in K^2_{(p,q+1)}(\Omega)$ where $0 \leq p \leq n$ and $1 \leq q \leq n-1$. Then there exist $g_j \in K^2_{(p,q)}(\Omega)$ for $1 \leq j \leq n$ such that*

$$g = \sum_{j=1}^n g_j \wedge d\overline{z}_j \text{ and } \sum_{j=1}^n \|g_j\| \lesssim \|g\|.$$

Proof. Let $0 \leq p \leq n, 1 \leq q \leq n-1$, and $f = \sum'_{|I|=p,|J|=q} f_{IJ} dz_I \wedge d\overline{z}_J = \overline{\partial}^* N_{p,q+1} g$. Then we can write

$$f = \sum_{j=1}^n f_j \wedge d\overline{z}_j$$

where f_j's are square integrable $(p,q-1)$-forms so that there are no common terms between $f_j \wedge d\overline{z}_j$ and $f_k \wedge d\overline{z}_k$ if $j \neq k$. This can be done as follows: Let $\vee$ denote the adjoint of the exterior multiplication. That is, if f is a (p,q)-form $d\overline{z}_j \vee f$ is a $(p,q-1)$-form such that $\langle h \wedge d\overline{z}_j, f\rangle = \langle h, d\overline{z}_j \vee f\rangle$ for all $h \in C^\infty_{(p,q-1)}(\mathbb{C}^n)$. Then we define

$$f_1 = d\overline{z}_1 \vee f$$
$$f_j = d\overline{z}_j \vee \left(f - \sum_{k=1}^{j-1} f_k \wedge d\overline{z}_k \right) \text{ for } j = 2, 3, \ldots, n.$$

Namely, f_1 is defined by collecting all terms that contain $d\overline{z}_1$ and writing their sum as $f_1 \wedge d\overline{z}_1$. Then we define f_2 by collecting the terms in $f - f_1 \wedge d\overline{z}_1$ with $d\overline{z}_2$ and writing their sum as $f_2 \wedge d\overline{z}_2$ etc. Since $\overline{\partial} g = 0$ and f is in the range of $\overline{\partial}^*$, we have $\overline{\partial} f = g$ and $\overline{\partial}^* f = 0$. So f is in the domains of $\overline{\partial}$ and $\overline{\partial}^*$. Also since f_j consists of terms f_{IJ} for some $|I| = p$ and $|J| = q$, "bar"

derivatives of f_j's come from "bar" derivatives of f. Then

$$\sum_{j,k=1}^{n}\left\|\frac{\partial f_j}{\partial \overline{z}_k}\right\| \lesssim \sum_{|I|=p,|J|=q}\sum_{k=1}^{n}\left\|\frac{\partial f_{IJ}}{\partial \overline{z}_k}\right\|.$$

This fact together with [Str10, Corollary 2.13] imply that

$$\sum_{j,k=1}^{n}\left\|\frac{\partial f_j}{\partial \overline{z}_k}\right\| \lesssim \|\overline{\partial} f\| + \|\overline{\partial}^* f\| = \|g\|.$$

Hence, $\|\overline{\partial} f_j\| \lesssim \|g\|$ for every j and

$$g = \overline{\partial} f = \sum_{j=1}^{n} \overline{\partial} f_j \wedge d\overline{z}_j.$$

Therefore, if we define $g_j = \overline{\partial} f_j$ we have $g = \sum_{j=1}^{n} g_j \wedge d\overline{z}_j$ and $\sum_{j=1}^{n}\|g_j\| \lesssim \|g\|$. □

Remark 3. The main idea for the proof of Theorem 1 depends on Lemma 1 and failure of Lemma 1 for $q = 0$ is the reason why Theorem 1 is still open for $q = 0$. One can see that Lemma 1 is not true for $q = 0$ as follows: Let $g = \phi(z_1)d\overline{z}_1$ where ϕ is a non-holomorphic function that is smooth on $\overline{\Omega}$. Then g is $\overline{\partial}$-closed but there is no holomorphic function g_1 such that $g = g_1 d\overline{z}_1$. Also it is interesting that the proof of Lemma 1 requires the existence of $(p, q-1)$-forms (that is, two form-levels below the starting form-level).

The following lemma shows that the converse of [Str10, Proposition 4.1] is true.

Lemma 2. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$, $n \geq 2$, and $0 \leq p \leq n, 1 \leq q \leq n-1$. Then $[P_{p,q}, \overline{z}_j]$ is compact on $K^2_{(p,q)}(\Omega)$ for all $1 \leq j \leq n$ if and only if $\overline{\partial}^* N_{p,q+1}$ is compact on $L^2_{(p,q+1)}(\Omega)$.*

Proof. We only need to prove one direction as compactness of $\overline{\partial}^* N_{p,q+1}$ on $L^2_{(p,q+1)}(\Omega)$ implies that $[P_{p,q}, \overline{z}_j]$ is compact on $\overline{\partial}$-closed (p,q)-forms for $1 \leq j \leq n$ (see [Str10, Proposition 4.1]).

To prove the other direction, assume that $\{g^k\}$ is a bounded sequence in $K^2_{p,q+1}(\Omega)$. Then Lemma 1 implies that for each k there exist $\overline{\partial}$-closed (p,q)-forms g^k_j for $1 \leq j \leq n$ such that $g^k = \sum_{j=1}^{n} g^k_j \wedge d\overline{z}_j$ and $\sum_{j=1}^{n}\|g^k_j\| \lesssim \|g^k\|$. Then we have

$$\overline{\partial}^* N_{p,q+1}(g^k) = (-1)^{p+q+1}\sum_{j=1}^{n}[P_{p,q}, \overline{z}_j](g^k_j).$$

Furthermore, if $[P_{p,q}, \overline{z}_j]$ is compact on $\overline{\partial}$-closed (p,q)-forms for $1 \leq j \leq n$, sequences $\{[P_{p,q}, \overline{z}_j](g^k_j)\}$ have convergent subsequences for each j. Hence $\overline{\partial}^* N_{p,q+1}$ is compact on $K^2_{(p,q+1)}(\Omega)$. On the other hand, compactness of $\overline{\partial}^* N_{p,q+1}$ on $\overline{\partial}$-closed forms is equivalent to compactness of $\overline{\partial}^* N_{p,q+1}$ on $L^2_{(p,q+1)}(\Omega)$ as $\overline{\partial}^* N_{p,q+1}$ vanishes on the orthogonal complement of $K^2_{(p,q+1)}(\Omega)$. □

Remark 4. It is interesting to observe that if $q = n-1$ then the commutators $[P_{p,n-1}, \bar{z}_j]$ are compact on $K^2_{(p,n-1)}(\Omega)$ on any bounded pseudoconvex domain Ω with sufficiently smooth boundary. This is a consequence of the fact that the $\bar{\partial}$-Neumann problem on (p,n)-forms is the classical Dirichlet problem. See [CS01, Remark after Corollary 5.1.7.])

Remark 5. It is still unknown if Lemma 2 is true when the commutators are restricted on $A^2_{(p,q)}(\Omega)$, the square integrable holomorphic (p,q)-forms, for $0 \leq p \leq n$, and $0 \leq q \leq n-1$. That is, it is not known whether compactness of $[P_{p,q}, \bar{z}_j]$ on $A^2_{(p,q)}(\Omega)$ for all j (equivalent to compactness of $\bar{\partial}^* N_{p,q+1}$ on $A^2_{(p,q+1)}(\Omega)$ by [Str10, Remark (ii) in Section 4.1]) imply compactness of $\bar{\partial}^* N_{p,q+1}$ on $K^2_{(p,q+1)}(\Omega)$.

Lemma 3. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n, n \geq 2, 0 \leq p \leq n$, and $1 \leq q \leq n-1$. Then compactness of $\bar{\partial}^* N_{p,q}$ on $K^2_{(p,q)}(\Omega)$ implies that $\bar{\partial}^* N_{p,q+1}$ is compact on $K^2_{(p,q+1)}(\Omega)$.*

Proof. Let $\{g^k\}$ be a bounded sequence of $\bar{\partial}$-closed $(p,q+1)$-forms. Then by Lemma 1 there exist $\bar{\partial}$-closed (p,q)-forms g^k_j's such that $g^k = \sum_{j=1}^n g^k_j \wedge d\bar{z}_j$ and $\sum_{j=1}^n \|g^k_j\| \lesssim \|g^k\|$.

Let us define $f^k = \sum_{j=1}^n \bar{\partial}^* N_{p,q}(g^k_j) \wedge d\bar{z}_j$. Then $\bar{\partial} f^k = g^k$ and compactness of $\bar{\partial}^* N_{p,q}$ implies that $\{f^k\}$ has a convergent subsequence. Therefore, $\bar{\partial}$ has a compact solution operator on $(p,q+1)$-forms. Hence, the canonical solution operator, $\bar{\partial}^* N_{p,q+1}$, is compact on $K^2_{(p,q+1)}(\Omega)$. $\square$

Remark 6. One corollary of Lemma 3 is the well known fact that compactness of $N_{p,q}$ implies compactness of $N_{p,q+1}$ (see [Str10, Proposition 4.5]). One can see this using Lemma 3 together with Range's formula ([Ran84]):

$$N_{p,q} = (\bar{\partial}^* N_{p,q})^* \bar{\partial}^* N_{p,q} + \bar{\partial}^* N_{p,q+1} (\bar{\partial}^* N_{p,q+1})^*.$$

Lemma 4. *Let Ω be a domain in $\mathbb{C}^n, 0 \leq p,q \leq n$ and $\phi, \psi \in L^\infty(\Omega)$. Assume that $[P_{p,q}, \phi]$ and $[P_{p,q}, \psi]$ are compact on $L^2_{(p,q)}(\Omega)$. Then $[P_{p,q}, \phi\psi]$ is compact on $L^2_{(p,q)}(\Omega)$.*

Proof. Assume that the operator $[P_{p,q}, \phi]$ and $[P_{p,q}, \psi]$ are compact on $L^2_{(p,q)}(\Omega)$. For $f \in L^2_{(p,q)}(\Omega)$ one can compute that

$$\begin{aligned} [P_{p,q}, \phi]\psi f + \phi[P_{p,q}, \psi] f &= (P_{p,q}\phi - \phi P_{p,q})\psi f + \phi(P_{p,q}\psi - \psi P_{p,q}) f \\ &= P_{p,q}\phi\psi f - \phi\psi P_{p,q} f \\ &= [P_{p,q}, \phi\psi] f. \end{aligned}$$

Hence,

$$[P_{p,q}, \phi\psi] f = [P_{p,q}, \phi]\psi f + \phi[P_{p,q}, \psi] f. \tag{1}$$

Therefore, $[P_{p,q}, \phi\psi]$ is compact on $L^2_{(p,q)}(\Omega)$ because the terms on the right hand side of (1) are composition of bounded operators with compact operators on $L^2_{(p,q)}(\Omega)$. $\square$

Let us define

$$\Gamma(p,q)=\{\phi\in C(\overline{\Omega}):[P_{p,q},\phi]:L^2_{(p,q)}(\Omega)\to L^2_{(p,q)}(\Omega)\text{ is compact}\}.$$

Corollary 1. *Let Ω be a bounded domain in $\mathbb{C}^n$. Then $\Gamma(p,q)$ is a C^* subalgebra (with identity) of $C(\overline{\Omega})$.*

Proof. The fact that compactness is preserved under the operator norm topology implies that $\Gamma(p,q)$ is a closed subspace of $C(\overline{\Omega})$. Lemma 4 implies that $\Gamma(p,q)$ is a subalgebra of $C(\overline{\Omega})$ and $1\in\Gamma(p,q)$ because $[P_{p,q},1]=0$. Finally, the fact that $\Gamma(p,q)$ is closed under conjugation follows from the formula $[P_{p,q},\phi]^*=-[P_{p,q},\overline{\phi}]$ and the fact that an operator is compact if and only if so is its adjoint. □

We note that Corollary 1 gives a characterization for compactness of $N_{p,q}$ for $q\geq 1$ in terms of C^* algebras. Namely, for $1\leq q\leq n-1$ the operator $N_{p,q+1}$ is compact if and only if $\Gamma(p,q)=C(\overline{\Omega})$ (compare to [Sal91]).

Proof of Theorem 1. We note that i. $\Rightarrow$ ii. is known (see [Str10, Proposition 4.2]) and ii. $\Rightarrow$ iii. is easy because $[P_{p,q},\overline{z}_j](f)=-\overline{\partial}^*N_{p,q+1}(d\overline{z}_j\wedge f)$ for $f\in K^2_{(p,q)}(\Omega)$.

Now we will prove iii. $\Rightarrow$ i.: Compactness of $[P_{p,q},\overline{z}_j]$ on $K^2_{(p,q)}(\Omega)$ for $1\leq j\leq n$, by Lemma 2, is equivalent to compactness of $\overline{\partial}^*N_{p,q+1}$ on $K^2_{(p,q+1)}(\Omega)$. On the other hand, if $q=n-1$ then compactness of $\overline{\partial}^*N_{p,q+1}$ is equivalent to compactness of $N_{p,q+1}$. In case $1\leq q\leq n-2$, Lemma 3 implies that compactness of $\overline{\partial}^*N_{p,q+1}$ on $K^2_{(p,q+1)}(\Omega)$ implies compactness of $\overline{\partial}^*N_{p,q+2}$ on $K^2_{(p,q+2)}(\Omega)$. Then [Str10, Proposition 4.2] implies that $N_{p,q+1}$ is compact on $L^2_{(p,q+1)}(\Omega)$.

The implication ii. $\Rightarrow$ iv. follows from the fact that $\overline{\partial}^*N_{p,q+1}$ vanishes on the orthogonal complement of $K^2_{(p,q+1)}(\Omega)$ and [Str10, Proposition 4.1].

We prove the implication iv. $\Rightarrow$ v. as follows: By Corollary 1 we know that $\Gamma(p,q)$ is a closed subalgebra with identity (of $C(\overline{\Omega})$) that is closed under conjugation. Then the assumption that the commutators $[P_{p,q},\overline{z}_j]$ are compact on $L^2_{(p,q)}(\Omega)$, for $1\leq j\leq n$, together with the Stone-Weierstrass Theorem imply that $\Gamma(p,q)=C(\overline{\Omega})$. That is $[P_{p,q},\phi]$ is compact for all $\phi\in C(\overline{\Omega})$.

Finally, the implications v. $\Rightarrow$ vi. and vi. $\Rightarrow$ iii. are obvious. □

Proof of Theorem2. If $q=n-1$ then $K^2_{(p,q+1)}(\Omega)=L^2_{(p,n)}(\Omega)$ and $[P_{p,q+1},\phi]$ is the zero operator, hence compact. So for the rest of the proof we may assume that $n\geq 3$ and $1\leq q\leq n-2$.

Let $g\in K^2_{(p,q+1)}(\Omega)$. Then Lemma 1 implies that there exist $g_j\in K^2_{(p,q)}(\Omega)$ for $1\leq j\leq n$ such that

$$g=\sum_{j=1}^n g_j\wedge d\overline{z}_j\text{ and }\sum_{j=1}^n\|g_j\|\lesssim\|g\|.$$

Now we will show that

$$[P_{p,q+1},\phi]g=(I-P_{p,q+1})\left(\sum_{j=1}^n([P_{p,q},\phi]g_j)\wedge d\overline{z}_j\right). \tag{2}$$

Since both sides of (2) are orthogonal to $K^2_{(p,q+1)}(\Omega)$ we only need to show that for any $h \in L^2_{(p,q+1)}(\Omega)$ that is orthogonal to $K^2_{(p,q+1)}(\Omega)$ we have

$$\left\langle [P_{p,q+1},\phi]g - (I - P_{p,q+1})\left(\sum_{j=1}^{n}([P_{p,q},\phi]g_j) \wedge d\overline{z}_j\right), h \right\rangle = 0$$

where $\langle .,. \rangle$ denotes the inner product on $L^2_{(p,q+1)}(\Omega)$. One can compute that

$$\begin{aligned}\left\langle [P_{p,q+1},\phi]g - (I - P_{p,q+1})\left(\sum_{j=1}^{n}([P_{p,q},\phi]g_j) \wedge d\overline{z}_j\right), h \right\rangle &= -\langle \phi g, h\rangle - \left\langle \sum_{j=1}^{n} P_{p,q}(\phi g_j)\wedge d\overline{z}_j, h\right\rangle \\ &\quad + \left\langle \sum_{j=1}^{n} \phi g_j \wedge d\overline{z}_j, h\right\rangle \\ &= -\left\langle \sum_{j=1}^{n} P_{p,q}(\phi g_j)\wedge d\overline{z}_j, h\right\rangle.\end{aligned}$$

The fact that $\overline{\partial}(f \wedge d\overline{z}_j) = (\overline{\partial} f) \wedge d\overline{z}_j$ implies that the $(p,q+1)$-forms $P_{p,q}(\phi g_j) \wedge d\overline{z}_j$ are $\overline{\partial}$-closed for $j = 1,\ldots,n$. Therefore, we have $\left\langle \sum_{j=1}^{n} P_{p,q}(\phi g_j)\wedge d\overline{z}_j, h\right\rangle = 0$ and the equality (2) is proven.

Let $\{g^k\} \subset K^2_{(p,q+1)}(\Omega)$ be a bounded sequence. Then Lemma 1 implies that for each k and $1 \le j \le n$ there exists $g^k_j \in K^2_{(p,q)}(\Omega)$ such that $g^k = \sum_{j=1}^{n} g^k_j \wedge d\overline{z}_j$ and $\sum_{j=1}^{n} \|g^k_j\| \lesssim \|g^k\|$. Furthermore, compactness of $[P_{p,q},\phi]$ on $K^2_{(p,q)}(\Omega)$ implies that for each $1 \le j \le n$ the sequence $\{[P_{p,q},\phi]g^k_j\}$ has a convergent subsequence. Now using (2) we conclude that the sequence $\{[P_{p,q+1},\phi]g^k\}$ has a convergent subsequence. Hence $[P_{p,q+1},\phi]$ is compact on $K^2_{(p,q+1)}(\Omega)$. □

Corollary 2. *Let Ω be a domain in $\mathbb{C}^n$, $0 \le p \le n$, $0 \le q \le n-1$, and $\phi \in C(\overline{\Omega})$. Assume that $[P_{p,q},\phi]$ be compact on $A^2_{(p,q)}(\Omega)$. Then $[P_{p,q+1},\phi]$ is compact on $A^2_{(p,q+1)}(\Omega)$.*

Proof. First of all, we note that (2) is valid on all domains, possibly unbounded or non-pseudoconvex. With this in mind, one can prove Corollary 2 the same way as Theorem 2 except, instead of Lemma 1 we need the following fact: Let $g \in A^2_{(p,q+1)}(\Omega)$, $0 \le p \le n$, and $0 \le q \le n-1$. Then for $1 \le j \le n$ there exists $g_j \in A^2_{(p,q)}(\Omega)$ such that $g = \sum_{j=1}^{n} g_j \wedge d\overline{z}_j$ and $\|g\|^2 = \sum_{j=1}^{n} \|g_j\|^2$. In fact, one can define $g_1 = d\overline{z}_1 \vee g$ and $g_j = d\overline{z}_j \vee \left(g - \sum_{k=1}^{j-1} g_k \wedge d\overline{z}_k\right)$ for $j = 2,3,\ldots,n$. □

Remark 7. Corollary 2 is stated for any domain in $\mathbb{C}^n$ and $0 \le q \le n-1$ while Theorem 2 is stated for bounded pseudoconvex domains and $1 \le q \le n-1$. Also Corollary 2 implies that on bounded pseudoconvex domains compactness of the $\overline{\partial}$-Neumann operator on square integrable holomorphic forms percolates up in the $\overline{\partial}$-complex. One can see this implication by observing that on a bounded pseudoconvex domain Ω compactness of the commutators $[P_{p,q},\phi]$ on $A^2_{(p,q)}(\Omega)$ for all $\phi \in C(\overline{\Omega})$ is equivalent to compactness of $N_{p,q+1}$ on $A^2_{(p,q+1)}(\Omega)$ (see, for example, [Str10, Remark (ii) in pg. 75]).

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